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GENERALIZATION OF JOUKOWSKI FORMULA TO AN AIRFOIL OF A CASCADE IN COMPRESSIBLE GAS STREAM WITH SUBSONIC VELOCITIES

By L. G. Loitsianskii

Translation

"Obobshchenie Formuly Zhukovskogo na Sluchai Profilia v Reshetke Obtekaemoi Szhimaemym Gazom pri Dozvukovykh Skorostiakh." Prikladnaia Matematika i Mekhanika. Vol. XIII; No. 2, 1949.

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GENERALIZATION OF JOUKOWSKI FORMULA TO AN AIRFOIL OF A CASCADE IN COMPRESSIBLE GAS STRFAM WITH SUBSONIC VELOCITIES*

By L. G. Loitsianskii

In the computation of the impellers of turbomachines, the Joukowski formula is applied; according to the formula, the lift force of the blade is equal to the product of the density of the gas, the mean vector of the velocities ahead of and behind the cascade, and the circulation about the blade.

The presence of high oncoming relative velocities of the gas flow at the blade make it necessary to account for the effect of the compressibility of the air on the lift force of the blade.

In 1934, Keldysh and Frankl (reference 1) showed that, in the case of the isolated wing profile, the Joukowski formula retains its usual form for both incompressible and compressible flow. As far as is known to this author, no generalizations have yet been made to the case of the airfoil in cascade.[1]

It is shown herein that, for subcritical values of the Mach number, the Joukowski formula may, with an approximation entirely sufficient for practical purposes, also be applied in its usual form to a cascade in a compressible flow, provided that the density is taken as the arithmetical mean of the densities of the gas at infinity ahead of and behind the cascade.

This simple result may be useful for the computation of cascades, especially in the solution of the problem of the resistance of a cascade when it is necessary to separate the lift force from the total force determined by the momentum theorem (reference 2).

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*"Obobshchenie Formuly Zhukovskogo na Sluchai Profilia v Reshetke Obtekaemoi Szhimaemym Gazom pri Dozvukovykh Skorostiakh." Prikladnaia Matematika i Mekhanika. Vol. XIII, No. 2, 1949, pp. 209-216.

[1]Very recently, there had come to this author's attention the as yet unpublished generalization of E. M. Berson, which differs from the one given herein.

1. Vector equation of lift force of an airfoil of a two-dimensional infinite cascade in an ideal incompressible fluid. - The flow about one of the airfoils of a cascade situated in a tube of flow having at infinity ahead of and behind the cascade the dimension equal to the pitch t in the direction of the axis of the cascade is considered (fig. 1). The density of the fluid is denoted by ρ, the velocities and pressures at infinity ahead of and behind the cascade by $\overline{w}_1$, $\overline{w}_2$ and p_1, p_2, respectively. When the momentum theorem is applied to the mass of fluid enclosed in the flow tube between the infinitely removed sections σ_1 and σ_2, the equation for the vector lift force may be written as

$$\overline{R} = (p_1 - p_2)\,\overline{t} + \rho\,(\overline{t} \cdot \overline{w}_1)\,\overline{w}_1 - \rho\,(\overline{t} \cdot \overline{w}_2)\,\overline{w}_2 \tag{1.1}$$

where the vector $\overline{t}$ is equal in magnitude to the pitch and is directed at right angles to the axis of the cascade downstream of the flow.

The difference in pressures $p_1 - p_2$ in equation (1.1) may be expressed in terms of the velocity vectors at infinity by the equation

$$p_1 - p_2 = \frac{\rho}{2}\,(w_2^2 - w_1^2) = \frac{\rho}{2}\,(\overline{w}_2 + \overline{w}_1) \cdot (\overline{w}_2 - \overline{w}_1) \tag{1.2}$$

The mean vector velocity $\overline{w}_m$ and the velocity $\overline{w}_d$ characterizing the vector change of velocity of the flow in passing through the cascade are then introduced:

$$\left.\begin{aligned} \overline{w}_m &= \frac{1}{2}\,(\overline{w}_1 + \overline{w}_2) \\ \overline{w}_d &= \overline{w}_2 - \overline{w}_1 \end{aligned}\right\} \tag{1.3}$$

Equation (1.2) then becomes

$$p_1 - p_2 = \rho\overline{w}_m \cdot \overline{w}_d \tag{1.4}$$

The condition of the conservation of mass along the tube of flow gives

$$\left.\begin{aligned} \bar{w}_1 \cdot \bar{t} = \bar{w}_2 \cdot \bar{t} = \bar{w}_m \cdot \bar{t} \\ \bar{w}_d \cdot \bar{t} = 0 \end{aligned}\right\} \tag{1.5}$$

From equations (1.4) and (1.5), equation (1.1) takes the form

$$\bar{R} = \rho\ (\bar{w}_m \cdot \bar{w}_d)\ \bar{t} - \rho\ (\bar{w}_m \cdot \bar{t})\ \bar{w}_d = \rho \bar{w}_m \times (\bar{t} \times \bar{w}_d) \tag{1.6}$$

As is seen from equation (1.6), the vector $\bar{R}$ lies in the flow plane and is directed perpendicular to $\bar{w}_m$ toward the side determined by the vector product on the right side of equation (1.6).

The magnitude of the vector $\bar{R}$ is given by

$$R = \rho w_m w_d t \tag{1.7}$$

2. Possible transformation of Bernoulli formula for compressible gas at subsonic velocities. - In the flow of an ideal compressible gas about a two-dimensional cascade, the condition of conservation of mass flow along a flow tube yields

$$\rho_1\ (\bar{t} \cdot \bar{w}_1) = \rho_2\ (\bar{t} \cdot \bar{w}_2) \tag{2.1}$$

and with the notation of equation (1.3), the equation for the vector lift force may be given in the form

$$\bar{R} = (p_1 - p_2)\ \bar{t} + \rho_1\ (\bar{t} \cdot \bar{w}_1)\ \bar{w}_1 - \rho_2\ (\bar{t} \cdot \bar{w}_2)\ \bar{w}_2 = (p_1 - p_2)\ \bar{t} - \rho_1\ (\bar{t} \cdot \bar{w}_1)\ \bar{w}_d \tag{2.2}$$

where ρ_1 and ρ_2 are the densities of the gas far ahead of and behind the cascade.

The first component of equation (2.2) is now considered. When it is assumed that the motion is adiabatic, the known relations for the pressure and the density are

$$p = p_0 \left(1 - \frac{k-1}{k+1} \lambda^2\right)^{\frac{k}{k-1}} = p_0 \left[1 - \frac{k}{k+1} \lambda^2 + \frac{k}{2(k+1)^2} \lambda^4 - . . .\right] \tag{2.3}$$

$$\rho = \rho_0 \left(1 - \frac{k-1}{k+1} \lambda^2\right)^{\frac{1}{k-1}} = \rho_0 \left[1 - \frac{1}{k+1} \lambda^2 + \frac{2-k}{2(k+1)^2} \lambda^4 - . . .\right] \tag{2.4}$$

where p_0 and ρ_0 denote the pressure and the density, respectively, of gas adiabatically brought to rest, k is the ratio of specific heats of gas at constant pressure c_p and constant volume c_v, and λ is the ratio of the velocity of flow w to the critical velocity of gas a_*, expressed in terms of the velocity of sound a_0 in the adiabatic stagnation of the gas:

$$\left.\begin{aligned} \lambda &= \frac{w}{a_*} \\ a_* &= \sqrt{\frac{2}{k+1}}\, a_0 = \sqrt{\frac{2k}{k+1} \frac{p_0}{\rho_0}} \end{aligned}\right\} \tag{2.5}$$

From expansion (2.3), the difference of the pressures becomes

$$p_1 - p_2 = \frac{k}{k+1} p_0(\lambda_2{}^2-\lambda_1{}^2) \left[1 - \frac{1}{2(k+1)} (\lambda_1{}^2+\lambda_2{}^2) + \epsilon_p\right] \tag{2.6}$$

where

$$\epsilon_p = \frac{1}{(\lambda_2{}^2-\lambda_1{}^2)k/(k+1)} \left[\left(1 - \frac{k-1}{k+1} \lambda_1{}^2\right)^{\frac{k}{k-1}} - \left(1 - \frac{k-1}{k+1} \lambda_2{}^2\right)^{\frac{k}{k-1}}\right] -$$

$$1 + \frac{1}{2(k+1)} (\lambda_1{}^2+\lambda_2{}^2) = \frac{2-k}{6(k+1)^2} (\lambda_1{}^4 + \lambda_1{}^2\lambda_2{}^2+\lambda_2{}^4) - . . . \tag{2.7}$$

and λ_1 and λ_2 are the values of the parameter λ ahead of and behind the cascade.

From equation (2.4), the arithmetic mean ρ_m of the densities ρ_1 and ρ_2 can be expressed as

$$\rho_m = \frac{1}{2}\left(\rho_1+\rho_2\right) = \rho_0\left[1 - \frac{1}{2(k+1)}\left(\lambda_1{}^2+\lambda_2{}^2\right) + \epsilon_\rho\right] \tag{2.8}$$

where

$$\epsilon_\rho = \frac{1}{2}\left[\left(1 - \frac{k-1}{k+1}\lambda_1{}^2\right)^{\frac{1}{k-1}} + \left(1 - \frac{k-1}{k+1}\lambda_2{}^2\right)^{\frac{1}{k-1}}\right] - 1 +$$

$$\frac{1}{2(k+1)}\left(\lambda_1{}^2+\lambda_2{}^2\right) = \frac{2-k}{4(k+1)^2}\left(\lambda_1{}^4+\lambda_2{}^4\right) - \ldots \tag{2.9}$$

When equations (2.6) and (2.8) are compared,

$$p_1 - p_2 = \frac{k}{k+1}\frac{p_0}{a_*{}^2}\left(w_2{}^2-w_1{}^2\right)\left(\frac{\rho_m}{\rho_0} + \epsilon_p - \epsilon_\rho\right)$$

$$= \rho_m \overline{w}_m \cdot \overline{w}_d\left[1 + \frac{\rho_0}{\rho_m}\left(\epsilon_p - \epsilon_\rho\right)\right] \tag{2.10}$$

Use of equations (2.5) and (1.3) yield, respectively,

$$\frac{k}{k+1}\frac{p_0}{\rho_0 a_*{}^2} = \frac{1}{2}\frac{kp_0}{\rho_0 a_0{}^2} = \frac{1}{2}$$

$$w_2{}^2 - w_1{}^2 = \frac{1}{2}\overline{w}_m \cdot \overline{w}_d$$

From equation (2.10) it follows that, with a certain error, the order of which will subsequently be discussed, the differences in pressures ahead of and behind the cascade may be determined by an equation similar to equation (1.4) for the incompressible fluid if the density of the gas is replaced by the arithmetic mean of the densities ahead of and behind the cascade.

3. Generalization of Joukowski formula to flow of a compressible gas at subsonic velocities about a cascade. - When equation (2.1) is used, the second component of equation (2.2) becomes

$$\rho_1(\bar{t}\cdot\bar{w}_1)\bar{w}_d = \left[\rho_m(\bar{t}\cdot\bar{w}_m) + \rho_1(\bar{t}\cdot\bar{w}_1) - \rho_m(\bar{t}\cdot\bar{w}_m)\right]\bar{w}_d$$

$$= \rho_m(\bar{t}\cdot\bar{w}_m)\bar{w}_d + \left[\frac{\rho_1\rho_2}{\rho_m}(\bar{t}\cdot\bar{w}_m) - \rho_m(\bar{t}\cdot\bar{w}_m)\right]\bar{w}_d$$

$$= \rho_m(\bar{t}\cdot\bar{w}_m)\bar{w}_d + \left(\frac{\rho_1\rho_2}{\rho_m} - \rho_m\right)(\bar{t}\cdot\bar{w}_m)\bar{w}_d$$

$$= \left(1 - \frac{\rho_m^{\ 2}-\rho_1\rho_2}{\rho_m^{\ 2}}\right)\rho_m(\bar{t}\cdot\bar{w}_m)\bar{w}_d$$

$$= \left[1 - \left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)^2\right]\rho_m(\bar{t}\cdot\bar{w}_m)\bar{w}_d$$

In order to estimate the second component in the brackets, note that, according to equation (2.4),

$$\left.\begin{aligned} \rho_1 - \rho_2 &= \frac{\rho_0}{k+1}(\lambda_2^{\ 2}-\lambda_1^{\ 2})\left[1 - \frac{2-k}{2(k+1)}(\lambda_1^{\ 2}+\lambda_2^{\ 2}) + \ldots\right] \\ \rho_1 + \rho_2 &= 2\rho_0\left[1 - \frac{1}{2(k+1)}(\lambda_1^{\ 2}+\lambda_2^{\ 2}) + \ldots\right] \end{aligned}\right\} \tag{3.1}$$

Hence,

$$\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)^2 = \frac{1}{4(k+1)^2}(\lambda_2^{\ 2}-\lambda_1^{\ 2})^2\left[1 + \frac{k-1}{k+1}(\lambda_1^{\ 2}+\lambda_2^{\ 2}) + \ldots\right] \tag{3.2}$$

Therefore,

$$\rho_1(\bar{t}\cdot\bar{w}_1)\bar{w}_d = \rho_m(\bar{t}\cdot\bar{w}_m)\bar{w}_d - \epsilon_\rho{}'\rho_m(\bar{t}\cdot\bar{w}_m)\bar{w}_d \tag{3.3}$$

where, from what has just been shown and when, for convenience, $\kappa = 1/(k-1)$,

$$\epsilon_\rho' = \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right)^2 = \frac{1}{4}\frac{(\rho_1-\rho_2)^2}{\rho_m^2}$$

$$= \left[\frac{\left[1 - (k-1)/(k+1)\lambda_1^2\right]^\kappa - \left[1 - (k-1)/(k+1)\lambda_2^2\right]^\kappa}{\left[1 - (k-1)/(k+1)\lambda_1^2\right]^\kappa - \left[1 - (k-1)/(k+1)\lambda_2^2\right]^\kappa}\right]^2$$

$$= \frac{1}{4(k+1)^2}(\lambda_2^2-\lambda_1^2)^2\left[1 + \frac{k-1}{k+1}(\lambda_1^2+\lambda_2^2) - . . .\right] \qquad (3.4)$$

The equation of the order of the error ϵ_ρ' will be ignored for the present; equation (2.2), on the basis of equations (2.10) and (3.3), may then be represented in the form

$$\overline{R} = \rho_m (\overline{w}_m \cdot \overline{w}_d)\,\overline{t} - \rho_m (\overline{t} \cdot \overline{w}_m)\,\overline{w}_d + \epsilon_R = \rho_m \overline{w}_m \times (\overline{t} \times \overline{w}_d) + \epsilon_R \qquad (3.5)$$

where ϵ_R represents a certain small vector

$$\epsilon_R = \frac{\rho_0}{\rho_m}(\epsilon_p - \epsilon_\rho)\,\rho_m (\overline{w}_m \cdot \overline{w}_d)\,\overline{t} + \epsilon_\rho'\rho_m (\overline{t} \cdot \overline{w}_m)\,\overline{w}_d \qquad (3.6)$$

the order of magnitude of which will subsequently be discussed.

When this small vector ϵ_R is neglected, generalization of the Joukowski formula to the flow of a compressible gas about an airfoil in cascade yields

$$\overline{R} = \rho_m \overline{w}_m \times (\overline{t} \times \overline{w}_d) \qquad (3.7)$$

When equation (3.7) is compared with the previously derived equation (1.6) for the flow of an incompressible fluid about a cascade, the following result is obtained: At subsonic velocities, the lift force of an airfoil in cascade can be approximately determined by the Joukowski formula for the incompressible fluid if the density of this fluid is equated to the arithmetic mean of the densities of the gas at a great distance ahead of and behind the cascade.

The absolute magnitude and the direction of the vector $\bar{R}$ with the assumed approximation will now be determined and the difference from the magnitude and the direction of the vector $\bar{R}$ determined for the incompressible fluid by equation (1.6) will be shown.

The vector product $\bar{t} \times \bar{w}_d$ is perpendicular to the plane of flow of the gas and therefore to the vector $\bar{w}_m$, which yields

$$R = \rho_m w_m \left| \bar{t} \times \bar{w}_d \right| \tag{3.8}$$

In contrast to the incompressible case, in the compressible case the vector $\bar{w}_d$ is not perpendicular to the vector $\bar{t}$, but is inclined by a certain small angle to the axis of the cascade (fig. 2). Hence, in contrast to the condition of perpendicularity (equation (1.5)) in the case of the incompressible gas, equation (2.1) gives

$$\bar{t} \cdot \bar{w}_d = \bar{t} \cdot (\bar{w}_2 - \bar{w}_1) = \bar{t} \cdot \left(\frac{1}{\rho_2} \rho_2 \bar{w}_2 - \frac{1}{\rho_1} \rho_1 \bar{w}_1 \right) = (\bar{t} \cdot \rho_1 \bar{w}_1) \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right)$$

$$\bar{t} \cdot \bar{w}_m = \frac{\bar{t}}{2} \cdot (\bar{w}_1 + \bar{w}_2) = \frac{\bar{t}}{2} \cdot \left(\frac{1}{\rho_1} \rho_1 \bar{w}_1 + \frac{1}{\rho_2} \rho_2 \bar{w}_2 \right) = \frac{1}{2} (\bar{t} \cdot \rho_1 \bar{w}_1) \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)$$

Hence,

$$\frac{\bar{t} \cdot \bar{w}_d}{\bar{t} \cdot \bar{w}_m} = 2 \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = \frac{\rho_1 - \rho_2}{\rho_m} = 2 \sqrt{\epsilon_\rho'} \tag{3.9}$$

Whence, from equation (3.2),

$$\bar{t} \cdot \bar{w}_d = 2 \sqrt{\epsilon_\rho'}\, \bar{t} \cdot \bar{w}_m = \frac{1}{k+1} (\lambda_2^2 - \lambda_1^2)(\bar{t} \cdot \bar{w}_m) \left[1 + \ldots \right] \tag{3.10}$$

Denoting the angles between the vector $\bar{t}$ and the vectors $\bar{w}_m$ and $\bar{w}_d$ by θ_m and θ_d yields the following expression for the modulus $|\bar{t} \times \bar{w}_d|$:

$$|\bar{t} \times \bar{w}_d| = tw_d \sin \theta_d = \sqrt{t^2 w_d{}^2 - t^2 w_d{}^2 \cos^2 \theta_d} = \sqrt{t^2 w_d{}^2 - (\bar{t} \cdot \bar{w}_d)^2}$$

$$= \sqrt{t^2 w_d{}^2 - 4\epsilon_\rho{}' (\bar{t} \cdot \bar{w}_m)^2} = tw_d \sqrt{1 - 4\epsilon_\rho{}' \left(\frac{w_m}{w_d}\right)^2 \cos^2 \theta_m}$$

$$= tw_d \left[1 - 2\epsilon_\rho{}' \left(\frac{w_m}{w_d}\right)^2 \cos^2 \theta_m\right] \tag{3.11}$$

The modulus of the vector $\bar{R}$ is thus given by

$$R = \rho_m w_m w_d t \left[1 - 2\epsilon_\rho{}' \left(\frac{w_m}{w_d}\right)^2 \cos^2 \theta_m\right] \approx \rho_m w_m w_d t \tag{3.12}$$

The direction of the vector $\bar{R}$ for compressible gas also remains perpendicular to the vector of the mean velocity $\bar{w}_m$ (fig. 2).

4. Estimate of accuracy of generalized Joukowski formula. - In order to answer the question of the possibility of application of the previously derived equations, it is necessary to estimate the order of the neglected terms and also the interval of Mach numbers in which, with a preassigned accuracy, the proposed approximate formula may be applied.

For this purpose, consider the estimate of the order of magnitude of the vector ϵ_R. The simplest method of estimating is applied in writing the evident inequality

$$\epsilon_R \leq \frac{\rho_0}{\rho_m} \left|\epsilon_p - \epsilon_\rho\right| \rho_m w_m w_d t \cdot \cos \theta_{md} + \epsilon_\rho{}' \rho_m w_m w_d t \cos \theta_m$$

or, when the cosines are replaced by their maximum values,

$$\epsilon_R < \left(\frac{\rho_0}{\rho_m} \left|\epsilon_p - \epsilon_\rho\right| + \epsilon_\rho{}'\right) \rho_m w_m w_d t \tag{4.1}$$

The ratio ρ_0/ρ_m is readily estimated from the known equations:

$$\rho_0 = \rho_1 \left(1 + \frac{k-1}{2} M_1^2\right)^\kappa = \rho_2 \left(1 + \frac{k-1}{2} M_2^2\right)^\kappa \qquad \kappa = \frac{1}{k-1}$$

Setting $M_1 = M_2 = 1$ yields

$$\rho_0 < \rho_1\left(1 + \frac{k-1}{2}\right)^\kappa = \rho_1 \left(\frac{k+1}{2}\right)^\kappa$$

$$\rho_0 < \rho_2\left(1 + \frac{k-1}{2}\right)^\kappa = \rho_2 \left(\frac{k+1}{2}\right)^\kappa$$

Assuming, however, that $M_1 = M_2 = 0$ yields

$$\rho_0 > \rho_1 \qquad \rho_0 > \rho_2$$

The result immediately follows:

$$1 < \frac{\rho_0}{\rho_m} < \left(\frac{k+1}{2}\right)^\kappa \qquad \kappa = \frac{1}{k-1} \tag{4.2}$$

The estimate of the remaining magnitudes entering the parentheses of equation (4.1) are now considered. For this purpose, the following simple device is used: The values $k = 4/3 = 1.33$ and $k = 3/2 = 1.5$, between which the region of most practical values of k is found (for air and certain other gases), are considered and for these values of k, the values of ϵ_p, ϵ_ρ, and ϵ_ρ' are computed. Similar computations may easily be made if desired for the interval $5/4 \leqslant k \leqslant 4/3$, and so forth.

For the chosen values of k, the exponents of the terms in equations (2.7), (2.9), and (3.4) become integers and the entire computation is easily carried out. Then, for $k = 4/3$,

$$\epsilon_p = \frac{\left(1 - \frac{1}{7}\lambda_1^2\right)^4 - \left(1 - \frac{1}{7}\lambda_2^2\right)^4}{\frac{4}{7}\left(\lambda_2^2 - \lambda_1^2\right)} - 1 + \frac{3}{14}\left(\lambda_1^2 + \lambda_2^2\right)$$

$$= \frac{1}{7^2}\left(\lambda_1^4 + \lambda_1^2\lambda_2^2 + \lambda_2^4\right) - \frac{1}{4 \cdot 7^3}\left(\lambda_1^6 + \lambda_1^4\lambda_2^2 + \lambda_1^2\lambda_2^4 + \lambda_2^6\right)$$

$$\epsilon_\rho = \frac{1}{2}\left[\left(1 - \frac{1}{7}\lambda_1^2\right)^3 + \left(1 - \frac{1}{7}\lambda_2^2\right)^3\right] - 1 + \frac{3}{14}\left(\lambda_1^2 + \lambda_2^2\right)$$

$$= \frac{3}{2 \cdot 7^2}\left(\lambda_1^4 + \lambda_2^4\right) - \frac{1}{2 \cdot 7^3}\left(\lambda_1^6 + \lambda_2^6\right)$$

$$\epsilon_\rho' = \frac{1}{4}\frac{\left(\rho_1 - \rho_2\right)^2}{\rho_m^2} = \frac{\rho_0^2}{4\rho_m^2}\left[\left(1 - \frac{1}{7}\lambda_1^2\right)^3 - \left(1 - \frac{1}{7}\lambda_2^2\right)^3\right]$$

$$= \frac{1}{4 \cdot 7^2}\left(\frac{\rho_0}{\rho_m}\right)^2\left(\lambda_1^2 - \lambda_2^2\right)^2\left[\left(1 - \frac{1}{7}\lambda_1^2\right)^2 + \right.$$

$$\left.\left(1 - \frac{1}{7}\lambda_1^2\right)\left(1 - \frac{1}{7}\lambda_2^2\right) + \left(1 - \frac{1}{7}\lambda_2^2\right)^2\right]^2$$

and for $k = 3/2$,

$$\epsilon_p = \frac{(1 - \frac{1}{5}\lambda_1^2)^3 - (1 - \frac{1}{5}\lambda_2^2)^3}{\frac{3}{5}(\lambda_2^2 - \lambda_1^2)} - 1 + \frac{1}{5}(\lambda_1^2 + \lambda_2^2)$$

$$= \frac{1}{3 \cdot 5^2}(\lambda_1^4 + \lambda_1^2\lambda_2^2 + \lambda_2^4)$$

$$\epsilon_\rho = \frac{1}{2}\left[\left(1 - \frac{1}{5}\lambda_1^2\right)^2 + \left(1 - \frac{1}{5}\lambda_2^2\right)^2\right] - 1 + \frac{1}{5}(\lambda_1^2 + \lambda_2^2)$$

$$= \frac{1}{2 \cdot 5^2}(\lambda_1^4 + \lambda_2^4)$$

$$\epsilon_\rho' = \frac{1}{4}\frac{(\rho_1 - \rho_2)^2}{\rho_m^2} = \frac{\rho_0^2}{4\rho_m^2}\left[\left(1 - \frac{1}{5}\lambda_1^2\right)^2 - \left(1 - \frac{1}{5}\lambda_2^2\right)^2\right]^2$$

$$= \frac{1}{4 \cdot 5^2}\left(\frac{\rho_0}{\rho_m}\right)^2(\lambda_1^2 - \lambda_2^2)^2\left[2 - \frac{1}{5}(\lambda_1^2 + \lambda_2^2)\right]^2$$

Therefore,

$$\epsilon_p - \epsilon_\rho = -\frac{1}{98}(\lambda_1^2 - \lambda_2^2)^2\left[1 - \frac{1}{14}(\lambda_1^2 + \lambda_2^2)\right] \quad \text{for} \quad k = \frac{4}{3} \tag{4.3}$$

$$\epsilon_p - \epsilon_\rho = -\frac{1}{150}(\lambda_1^2 - \lambda_2^2)^2 \quad \text{for} \quad k = \frac{3}{2} \tag{4.4}$$

On the basis of the obtained formulas, an estimate of the expression

$$\frac{\epsilon_R}{\rho_m w_m w_d t} = \frac{\rho_0}{\rho_m}\left|\epsilon_p - \epsilon_\rho\right| + \epsilon_\rho' \tag{4.5}$$

for any values of k in the interval $4/3 \leq k \leq 3/2$ is given: From equations (5.2), (5.3) [Ed. note: equations (4.2) and (4.3)], and so forth, for $k = 4/3$,

$$\left[\frac{1}{98}\,\frac{6}{7} + \frac{1}{4 \cdot 7^2}\, 3^2 \left(\frac{6}{7}\right)^4\right] (\lambda_1^2 - \lambda_2^2)^2 < \frac{\rho_0}{\rho_m}\left|\epsilon_p - \epsilon_\rho\right| + \epsilon_\rho'$$

$$< \left[\left(\frac{7}{6}\right)^3 \frac{1}{98} + \frac{1}{4 \cdot 7^2}\left(\frac{7}{6}\right)^6 9\right] (\lambda_1^2 - \lambda_2^2)^2$$

or

$$0.033\ (\lambda_1^2 - \lambda_2^2)^2 \leq \frac{\rho_0}{\rho_m}\left|\epsilon_p - \epsilon_\rho\right| + \epsilon_\rho' < 0.133\ (\lambda_1^2 - \lambda_2^2)^2 \tag{4.6}$$

and for $k = 3/2$,

$$\left[\frac{1}{150} + \frac{1}{4 \cdot 5^2}\left(\frac{8}{5}\right)^2\right] (\lambda_1^2 - \lambda_2^2)^2 < \frac{\rho_0}{\rho_m}\left|\epsilon_p - \epsilon_\rho\right| + \epsilon_\rho'$$

$$< \left[\left(\frac{5}{4}\right)^2 \frac{1}{150} + \frac{1}{4 \cdot 5^2}\left(\frac{5}{4}\right)^4 4\right] (\lambda_1^2 - \lambda_2^2)^2$$

or

$$0.032\ (\lambda_1^2 - \lambda_2^2)^2 < \frac{\rho_0}{\rho_m}\left|\epsilon_p - \epsilon_p\right| + \epsilon_p' < 0.108\ (\lambda_1^2 - \lambda_2^2)^2 \tag{4.7}$$

Thus, for the magnitude ϵ_R, the estimate is

$$0.033\ (\lambda_1^2 - \lambda_2^2)^2 < \frac{\epsilon_R}{\rho_m w_m w_d t} < 0.133\ (\lambda_1^2 - \lambda_2^2)^2 \tag{4.8}$$

From the parameters λ_1 and λ_2, the Mach numbers $M_1 = w_1/a_1$ and $M_2 = w_2/a_2$ are obtained, where a_1 and a_2 are the velocities ahead of and behind the cascade. By the known equations of gas dynamics,

$$\lambda_1^2 = \frac{k+1}{2} \frac{M_1^2}{1 + \frac{1}{2}(k-1) M_1^2}$$

$$\lambda_2^2 = \frac{k+1}{2} \frac{M_2^2}{1 + \frac{1}{2}(k-1) M_2^2}$$

Whence,

$$\lambda_1^2 - \lambda_2^2 = \frac{k+1}{2} \frac{M_1^2 - M_2^2}{\left[1 + \frac{1}{2}(k-1) M_1^2\right]\left[1 + \frac{1}{2}(k-1) M_2^2\right]}$$

Hence,

$$\frac{2}{k+1} (M_1^2 - M_2^2)^2 < (\lambda_1^2 - \lambda_2^2)^2 < \left(\frac{k+1}{2}\right)^2 (M_1^2 - M_2^2)^2 \qquad (4.9)$$

Thus, in the interval $4/3 \leq k \leq 3/2$, the following estimate may be used in place of equation (4.8):

$$0.03 \, (M_1^2 - M_2^2)^2 < \frac{\epsilon_R}{\rho_m w_m w_m t} < 0.2 \, (M_1^2 - M_2^2)^2 \qquad (4.10)$$

Equation (4.10) is a very rough estimate; in fact, the modulus of the vector ϵ_R will be much less. From this simple estimate, however, the possibility of practical application of the approximate equation (3.7) is evident. Thus, for example, if at the entrance to a turbine cascade the Mach number is equal to $M_1 = 0.2$ and at the exit $M_2 = 0.7$, the relative error will not exceed 4 percent, even for the very rough estimate assumed.

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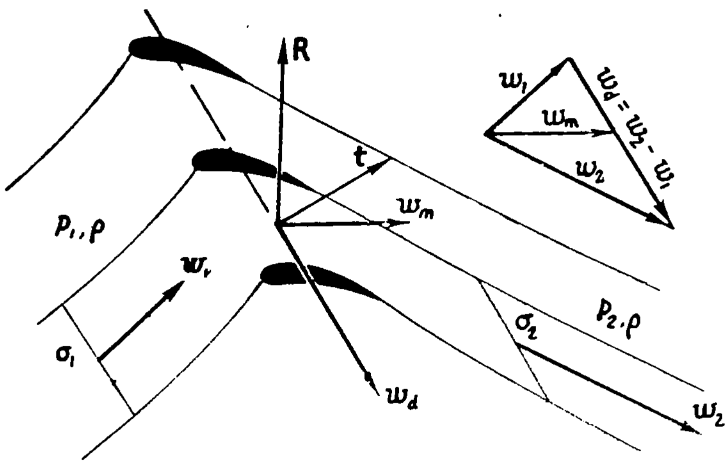

Figure 1.

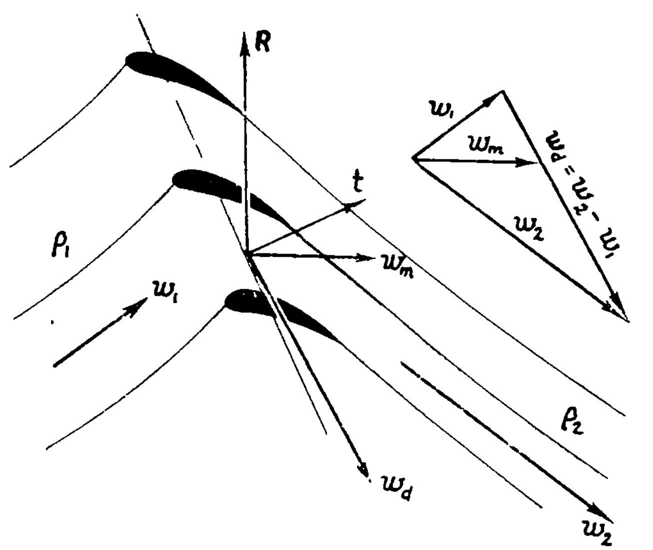

Figure 2.